

# COMPUTABILITY OF FILTERED QUANTITIES FOR THE BURGERS' EQUATION

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**Abstract.** We prove a priori and a posteriori error estimates for filtered solutions of the Burgers' equation discretized using two different stabilized finite element method, linear artificial viscosity and nonlinear artificial viscosity. We show that the error of filtered quantities converges with an order in  $h$ . The constant of the estimate is independent of both the regularity of the exact solution and the Reynolds number, it is moderate and primarily depending on the smoothness of the initial data. The convergence order and the norm in which convergence is measured are associated to the width of the filter. Similar estimates for the unfiltered quantities on the other hand lead to the huge exponential factors that are known to make estimates deteriorate for high Reynolds flows.

**1. Introduction.** The computability of solutions to the equations of fluid mechanics remains an open problem. Nonlinear effects such as turbulence and shocks make the stability analysis of the equations very difficult and any error estimate will include constants that are huge, making the estimates worthless in practice. Even in the simple case of the Burgers' equation it appears that the  $L^2$ -error analysis for finite element methods is incomplete, indeed the perturbation equation of the Burgers' equation does not seem to be stable in the  $L^2$ -norm. The objective of the present work is to show that for finite element discretizations of the Burgers' equation, filtered quantities are provably computable, provided sufficient artificial viscosity is added to make the stability properties of the continuous equation carry over to the discrete case. The computability of the corresponding unfiltered quantities appears difficult to establish theoretically, due to the huge stability factors involved.

We will mainly be interested in computations where the mesh Reynolds number is much larger than one. The Reynolds number in this context is a characteristic velocity  $U$ , to be defined, times the smallest length scale of the flow (i.e. the mesh size  $h$ ) divided by the kinematic velocity  $\nu$ :

$$Re := \frac{Uh}{2\nu}.$$

Energy conservative methods, such as the standard Galerkin finite element method, have poor stability properties in this regime due to accumulation of fine scale energy on the resolved coarse scales. Stabilized finite element methods, on the other hand, have some properties that make them particularly interesting for the solution of high Reynolds flow problems. It has for instance been proven that stabilized finite element methods applied to linear advection–diffusion problems have optimal convergence away from layers with a constant independent of the exact solution in the layer region [16, 7]. The stabilization also allows to improve the global convergence order for the incompressible Navier-Stokes' equations by a factor  $h^{\frac{1}{2}}$  in estimates independent of  $\nu$ , provided the solution is smooth [6]. Long time stability results have been obtained for the subscales in the variational multiscale method suggested in [1]. It has also been argued heuristically that the dissipation induced by stabilization matches the decay rate of the power spectrum under mesh refinement in the high Reynolds regime [3, 22]. Finally there are results proving that for the Burgers' equation, provided the stabilization gives sufficient control of the discrete solution, the asymptotic limit of the sequence of discrete solutions is the entropy solution [18, 4].

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All these results are however of mainly qualitative nature in the nonlinear case since they do not give any information on what one actually can compute on a given scale. The error estimates are of purely academic interest since they will either ask for a regularity of the solution that may not hold, or they feature a multiplicative factor such as  $\exp(\|\nabla u\|_{L^\infty} T)$  or  $\exp(\nu^{-1} T)$  where usually  $\|\nabla u\|_{L^\infty}$  is very big and  $\nu$  very small in the regime that interests us. Asymptotic estimates for the hyperbolic Burgers equation on the other hand give no information of the accuracy of a certain computation with a fixed  $h$  and is therefore also of reduced interest in practice.

We will consider two different stabilized finite element methods in this work. First the classical linear artificial viscosity resulting in a first order scheme will be introduced to the standard Galerkin method and then a nonlinear viscosity in the spirit of that proposed in [4]. This is a shockcapturing technique related to those proposed in [18] or the entropy viscosity of [13].

Consider the simple model case of the Burgers' equation with periodic boundary conditions, on the space-time domain  $Q := I \times (0, T)$ , with  $I := (0, 1)$

$$\begin{aligned} \partial_t u + \frac{1}{2} \partial_x u^2 - \nu \partial_{xx} u &= 0 \text{ in } Q \\ u(0, t) &= u(1, t) \text{ for } t \in (0, T) \\ u(x, 0) &= u_0(x) \text{ for } x \in I. \end{aligned} \tag{1.1}$$

For the discretization we use the simplest of stabilized finite element methods, piecewise affine continuous approximation with linear, or nonlinear, artificial viscosity. In this framework we prove estimates for a regularized error. The interest of these estimates stems from the fact that the constant of the estimates are of moderate size and only depends on the regularity of the initial data, hence there is no dependence on the Reynolds number, nor of the global regularity of the solution. Error estimates for averaged or postprocessed quantities have already been proposed in the literature, see for instance [10, 15, 12], but to the best of our knowledge they all suffer from the huge exponential constants, or dependence on the Reynolds number, discussed above.

First we discuss the  $L^\infty(0, T; L^2(I))$  stability of the Burgers' equation and conclude that the resulting estimate includes an exponential factor of the type

$$\exp(\|\partial_x u\|_{L^\infty} T)$$

reflecting a possible instability in the  $L^2$ -norm. Then we introduce the finite element discretization and discuss the stability properties of the method in some detail. Finally we consider filtering of the final solution and show that the perturbation equation corresponding to the filtered solution has improved stability properties and the error may therefore be upper bounded independently of both the regularity of the exact solution and the physical viscosity. The estimates also give a precise rate of convergence in  $h$ , depending only on the filter width. Typically we will consider the following differential filter known from time relaxation methods (see for instance [8, 20])

$$-\delta^2 \Delta \tilde{u} + \tilde{u} = u(\cdot, T) \quad \text{on } I \tag{1.2}$$

with  $\tilde{u}(0) = \tilde{u}(1)$  and  $\delta$  the filter width. As we shall see, although  $\|(u - u_h)(\cdot, T)\|_{L^2(I)}$ , where  $u_h$  denotes the finite element approximation of (1.1), does not appear to allow for error estimates with moderate constants,  $\|(\tilde{u} - \tilde{u}_h)(\cdot, T)\|_{L^2(I)}$  does. Indeed, in

the high Reynolds number regime we prove the error estimate

$$\|\delta\partial_x(\tilde{u} - u)(T)\|_{L^2(I)} + \|(\tilde{u} - \tilde{u}_h)(T)\|_{L^2(I)} \leq \tilde{C}(u_0, T) \exp(D_0 T) \left(\frac{h}{\delta^2}\right)^{\frac{1}{2}} \quad (1.3)$$

where  $\tilde{u}$  and  $\tilde{u}_h$  are the filtered exact and computational solution respectively. The constant in (1.3) depend only on the initial data, the mesh geometry and the final time. We will use the notation  $U_0 := \sup_{x \in I} |\pi_h u_0(x)|$  and  $D_0 := \sup_{x \in I} \frac{1}{2} \partial_x(u_0 + \pi_h u_0(x))$ , where  $\pi_h$  denotes the  $L^2$ -projection onto the finite element space. We will choose  $u_0$  as a smooth function and by the stability of the  $L^2$ -projection on regular meshes we have  $U_0 \lesssim \sup_{x \in I} |u_0(x)|$  and  $D_0 \lesssim \sup_{x \in I} |\partial_x u_0(x)|$ , so that estimate depending on  $U_0$  and  $D_0$  are indeed mesh independent. For simplicity we assume  $u_0 \in C^\infty(I)$ , this does not exclude the formation of sharp layers with gradients of order  $\nu^{-1}$  at later times. For fixed filter width (1.3) results in a convergence rate of order  $h^{\frac{1}{2}}$ . If on the other hand the filter width is related to the mesh size  $\delta \sim h^\alpha$  with  $\alpha < \frac{1}{2}$  we get the convergence rate  $h^{\frac{1-2\alpha}{2}}$ . The parameter  $\delta$  determines how strong the localization of the norm is. The choice  $\delta = 1$  leads to a norm related to the  $H^{-1}$ -norm and the choice  $\delta = h$  leads to a norm similar to the  $L^2$ -norm. Clearly the estimates proposed here only makes sense for  $0 \leq \alpha < \frac{1}{2}$ . Showing that no error bounds in a norm similar to the  $L^2$  case can be obtained. If the aim of the computation is to use  $\tilde{u}_h$  as an approximation of  $u$ , we observe that the  $L^2$ -norm perturbation error induced by the filtering in the smooth portion  $S$  of  $I$  satisfies

$$\|u - \tilde{u}\|_{L^2(S)} \lesssim \delta^2.$$

Here we use the notation  $a \lesssim b$  defined by  $a \leq Cb$  with  $C$  a constant independent of  $h$ , the physical parameters and of the exact solution.

The computational error of the filtered solution, as an approximation of  $u$  in the smooth part of the domain on the other hand is therefore given by

$$\|u - \tilde{u}_h\|_{L^2(S)} \leq \|u - \tilde{u}\|_{L^2(S)} + \|\tilde{u} - \tilde{u}_h\|_{L^2(S)} \lesssim \delta^2 + \left(\frac{h}{\delta^2}\right)^{\frac{1}{2}}$$

and with  $\delta := h^\alpha$  we obtain

$$\|u - \tilde{u}_h\|_{L^2(S)} \lesssim h^{2\alpha} + h^{\frac{1-2\alpha}{2}}.$$

It follows that the optimal convergence order away from layers is obtained for  $\alpha = \frac{1}{6}$ , leading to the  $L^2$ -error  $\mathcal{O}(h^{\frac{1}{3}})$  in the smooth part of the domain  $S$ . It is also worth noting that thanks to the maximum principle, under the reasonable assumption that  $\text{meas}(I \setminus S) = \mathcal{O}(\mu/U_0)$  we have the following bound for the global  $L^2$ -error

$$\|u - \tilde{u}_h\|_{L^2(I)} \leq \|u - \tilde{u}_h\|_{L^2(S)} + U_0 \left(\frac{\max(\mu, U_0 h)}{U_0}\right)^{\frac{1}{2}} \lesssim h^{\frac{1}{3}} + (U_0 \max(\mu, U_0 h))^{\frac{1}{2}}$$

which implies that for  $Re \gg 1$  the global convergence order in  $L^2$  is of order  $\mathcal{O}(h^{\frac{1}{3}})$ . Due to the compound effect of the regularization error and the discretization error, most likely this estimate is sharp for  $\tilde{u}_h$ , whereas  $u_h$  could provide a superior approximation of  $u$ .

The derivation of the estimate (1.3) uses:

- sharp energy stability estimates for the finite element method,
- maximum principles for the finite element solution and its first derivative,
- a priori stability estimates on a linearized dual problem with regularized data,
- Galerkin orthogonality and approximability.

The present analysis draws on previous ideas on a posteriori error analysis for hyperbolic problems and computability in CFD, mainly from [17], [2] and [14]. Below we are interested in estimates where we have precise control of all constants. This implies controlling the asymptotic growth of the residual and working with the exact dual adjoint, involving both the approximate and the exact solution. In the literature, the adjoint equation is often linearized around the exact solution only, eliminating the feedback of the stability properties of the numerical scheme in the sensitivity analysis and preventing the derivation of error estimates.

The viscosity coefficient of the dual problem herein is the physical viscosity and not the artificial viscosity, as often has been suggested. This way the dual problem contains the accurate sensibility information of the physical problem.

**2. The Burgers' equation with dissipation.** The wellposedness of the equation (1.1) for  $\nu \geq 0$  is well known (see for instance [21]) it is also known that for  $\nu > 0$  by parabolic regularization the solution is  $C^\infty(I)$ . This high regularity however does not necessarily help us when approximating the solution, since we are interested in computations using a mesh-size that is much larger than the viscosity and still want the bounds to be independent of high order Sobolev norms of the exact solutions and of  $\nu$ .

**2.1.  $L^2$ -stability of Burgers' equation.** Consider a general perturbation  $\eta(x)$  of the initial data of (1.1).

$$\begin{aligned} \partial_t \hat{u} + \frac{1}{2} \partial_x \hat{u}^2 - \nu \partial_{xx} \hat{u} &= 0 \text{ in } Q \\ \hat{u}(0, t) &= \hat{u}(1, t) \text{ for } t \in (0, T) \\ \hat{u}(x, 0) &= u_0(x) + \eta(x) \text{ for } x \in I. \end{aligned} \tag{2.1}$$

Taking the difference of (2.1) and (1.1) leads to the perturbation equation for  $\hat{e} := \hat{u} - u$  with  $a(u, \hat{u}) := \frac{1}{2}(u + \hat{u})$ ,

$$\begin{aligned} \partial_t \hat{e} + \partial_x (a(u, \hat{u}) \hat{e}) - \nu \partial_{xx} \hat{e} &= 0 \text{ in } Q, \\ \hat{e}(0, t) &= \hat{e}(1, t) \text{ for } t \in (0, T) \\ \hat{e}(x, 0) &= \eta(x) \text{ for } x \in I. \end{aligned} \tag{2.2}$$

Multiplying equation (2.2) by  $\hat{e}$  and integrating over  $Q$  leads to the energy equality

$$\frac{1}{2} \|\hat{e}(T)\|_{L^2(I)}^2 + \|\nu^{\frac{1}{2}} \partial_x \hat{e}\|_{L^2(Q)}^2 = \frac{1}{2} \|\eta\|_{L^2(I)}^2 - \int_Q (\partial_x a(u, \hat{u})) \hat{e}^2.$$

We know that due to shock formation  $-\partial_x a(u, \hat{u}) \sim \nu^{-1}$ . Any attempt to obtain control of  $\|\hat{e}(T)\|_{L^2(\Omega)}^2$  in terms of the initial data will rely on Gronwall's lemma, leading to

$$\|\hat{e}(T)\|_{L^2(I)}^2 \leq C_a \|\eta\|_{L^2(I)}^2$$

with the exponential factor

$$C_a := \exp(\|\partial_x a(u, \hat{u})\|_{L^\infty(Q)} T) \sim \exp(T/\nu),$$

which is exactly the bad factor discussed in the introduction. It tells us that we have stability (and hence computability) only up to the formation of shocks. Using this type of argument in the analysis of the finite element method leads to error estimates of the type derived in [6], useful only for solutions with moderate gradients.

**2.2. Maximum principles for Burgers' equation.** It is well known that the equation (1.1) satisfies a maximum principle on the form:

$$\sup_{(x,t) \in Q} |u(x,t)| \leq \sup_{x \in I} |u_0(x)|. \quad (2.3)$$

This follows using the smoothness of the solution  $u$  and the standard technique of assuming a local extremum in a point and showing that  $\partial_t u$  has the appropriate sign in that point (or in the hyperbolic case, using the method of characteristics). For our purposes we also need some precise information on the derivative. Since the solution of (1.1) is smooth we may derive the equation in space to obtain the following equation for the space derivative  $w := \partial_x u$ :

$$\begin{aligned} \partial_t w + u \partial_x w - \nu \partial_{xx} w &= -w^2 \text{ in } Q \\ w(0, t) &= w(1, t) \text{ for } t \in (0, T) \\ w(x, 0) &= \partial_x u_0(x) \text{ for } x \in I. \end{aligned} \quad (2.4)$$

Assuming that  $w$  takes its maximum in some point  $x \in I$  and noting that  $\partial_x w(x) = 0$  and  $\partial_{xx} w(x) < 0$  it follows that  $\partial_t w < 0$  at the maximum and we deduce the bound:

$$\max_{(x,t) \in Q} \partial_x u \leq \max_{x \in I} \partial_x u_0. \quad (2.5)$$

It follows by the smoothness of the initial data that the space derivative is bounded above for all times.

**3. Artificial viscosity finite element method.** Discretize the interval  $I$  with  $N$  elements and let the local mesh-size be defined by  $h := 1/N$ . We denote the computational nodes by  $x_i := i h$ ,  $i = 0, \dots, N$ , defining the elements  $I_j := [x_j, x_{j+1}]$ ,  $j = 0, \dots, N-1$ , and the standard nodal basis functions  $\{v_i\}_{i=0}^N$ , such that  $v_i(x_j) = \delta_{ij}$ , with  $\delta_{ij}$  the Kronecker delta. To impose periodic boundary conditions we identify the node  $x_0$  with  $x_N$  and define the corresponding basis function  $v_{0N} : (x_0, x_1) \cup (x_{N-1}, x_N) \mapsto \mathbb{R}$  by  $v_0$  on  $(x_0, x_1)$  and by  $v_N$  on  $(x_{N-1}, x_N)$ . This basis function then replaces  $v_0$  and  $v_N$ , leading to a total of  $N$  degrees of freedom. For simplicity we use the notation  $v_0$  for the basis function  $v_{0N}$ . The finite element space is given by

$$V_h := \left\{ \sum_{i=0}^{N-1} u_i v_i, \text{ where } \{u_i\}_{i=0}^{N-1} \in \mathbb{R}^N \right\}.$$

We define the standard  $L^2$  inner product on  $X \subset I$  by

$$(v_h, w_h)_X := \int_X v_h w_h \, dx.$$

The discrete form corresponding to mass-lumping reads

$$(v_h, w_h)_h := \sum_{i=0}^{N-1} v_h(x_i) w_h(x_i) h.$$

The associated norms are defined by  $\|v\|_X := (v, v)_X^{\frac{1}{2}}$ , for all  $v \in L^2(X)$ , if  $X$  coincides with  $I$  the subscript is dropped, and  $\|v_h\|_h := (v_h, v_h)_h^{\frac{1}{2}}$  for all  $v_h \in V_h$ . Note that, by norm equivalence on discrete spaces, for all  $v_h \in V_h$  there holds

$$\|v_h\|_h \lesssim \|v_h\| \lesssim \|v_h\|_h.$$

Using the above notation the artificial viscosity finite element space semi-discretization of (1.1) writes, given  $u_0 \in C^\infty(\Omega)$  find  $u_h(t) \in V_h$  such that  $(u_h(0), v_h)_I = (u_0, v_h)_I$  and

$$(\partial_t u_h, v_h)_h + \left( \partial_x \frac{u_h^2}{2}, v_h \right)_I + (\hat{\nu} \partial_x u_h, \partial_x v_h)_I = 0, \text{ for all } v_h \in V_h \text{ and } t > 0, \quad (3.1)$$

where we propose two different forms of  $\hat{\nu}$ :

1. linear artificial viscosity:

$$\hat{\nu} := \max(U_0 h / 2, \nu); \quad (3.2)$$

2. nonlinear artificial viscosity:

Let  $0 \leq \epsilon$  and

$$\nu_0(u_h)|_{I_i} := \frac{1}{2} \|u_h\|_{L^\infty(I_i)} \max_{x \in \{x_i, x_{i+1}\}} \frac{|[\partial_x u_h]|_x|}{2\{|\partial_x u_h|\}_x + \epsilon}, \quad (3.3)$$

where  $[\partial_x u_h]|_{x_i}$  denotes the jump of  $\partial_x u_h$  over the node  $x_i$  and  $\{|\partial_x u_h|\}_{x_i}$  denotes the average of  $|\partial_x u_h|$  over  $x_i$ . If  $\epsilon = 0$  and  $\{|\partial_x u_h|\}_{x_i} = 0$  we replace the quotient  $|[\partial_x u_h]|_{x_i} / \{|\partial_x u_h|\}_{x_i}$  by zero.

Further let

$$\xi(u_h)|_{I_i} := \begin{cases} 1 & \text{if } \partial_x u_h|_{I_i} > 0, \partial_x u_h|_{I_i} > \partial_x u_h|_{I_{i+1}} > 0 \\ & \text{and } \partial_x u_h|_{I_i} \geq \partial_x u_h|_{I_{i-1}} > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\nu_1(u_h)|_{I_i} := \xi(u_h)|_{I_i} \max\left(\nu_0|_{I_{i-1}} \frac{\partial_x u_h|_{I_{i-1}}}{\partial_x u_h|_{I_i}}, \nu_0|_{I_{i+1}} \frac{\partial_x u_h|_{I_{i+1}}}{\partial_x u_h|_{I_i}}\right). \quad (3.4)$$

Finally define:

$$\hat{\nu}(u_h)|_{I_i} := \max(\nu, h(\nu_0|_{I_i} + \nu_1|_{I_i})). \quad (3.5)$$

The rationale for the nonlinear viscosity is to add first order viscosity at local extrema of the solution  $u_h$  so that (2.3) holds also for the discrete solution and enough viscosity at positive extrema of  $\partial_x u_h$ , making (2.5) carry over to the discrete setting. The most important term is  $\nu_0$ , ensuring the discrete maximum principle. The other part  $\nu_1$  is merely a correction that ensures that the viscosity at local maxima of  $\partial_x u_h$  dominates that of the surrounding elements. Indeed the role of the function  $\xi(u_h)$  is to act as

an indicator function for the elements where the local maxima of  $\partial_x u_h$  are taken and modify the viscosity there. By construction, if  $\xi(u_h) = 1$  in one element it must be zero in the neighbouring elements. Note that formally  $\hat{\nu}(u_h) \approx O(\max(U_0 h^{3/2}, \nu))$  in the smooth part of the solution, so that in principle we can expect higher order convergence away from local extrema. Note that the perturbation  $\nu_1$  close to extrema of the gradient of the solution has no impact on the formal order. The high order convergence properties and the effect of the regularization parameter  $\epsilon$  will be explored in the numerical section.

*Remark 1. Note that in the linear case the viscosity  $\hat{\nu}$  may be written as*

$$\hat{\nu} := \nu \max(1, Re_h), \quad \text{with } Re_h := \frac{U_0 h}{2\nu}$$

*reflecting that in the high Reynolds number regime the viscosity is increased artificially to be order  $h$ .*

**3.1. Existence of solution to the semidiscretized system.** For the linear method existence and uniqueness of solutions of (3.1) follows using standard methods. First we observe that the nonlinear function of the dynamical system is locally Lipschitz and then anticipating the global upper bound (3.6) we conclude that a global solution exists and is unique.

The nonlinear method obtained when (3.1) is used with the viscosity (3.5) results in a dynamical system with discontinuous righthand side (even for  $\epsilon > 0$  the contribution from  $\nu_1$  introduces discontinuities). Existence of solutions to (3.1) with the nonlinear viscosity (3.5) is obtained using Filippov theory [11]. Anticipating the results of the next section, we may conclude that a solution exists, since by the discrete maximum principle (3.6), for fixed  $h$ , there exists  $M > 0$  such that

$$\left| \left( \partial_x \frac{u_h^2}{2}, v_i \right)_I + (\hat{\nu} \partial_x u_h, \partial_x v_i)_I \right| \leq M$$

and hence  $|\partial_t u_h(x, t)| < M$  for all  $(x, t) \in Q$ . The question of uniqueness is more involved, but using the following argument we conjecture that the solution is forward unique. Observe that in a given pair of neighbouring cells  $\Delta I := I_{i-1} \cup I_i$ , the system can either have  $\xi = 0$  in both cells, or have  $\xi = 0$  in one of the cells and have  $\xi = 1$  in the other. It follows that there are three states possible in  $\Delta I$  across the boundaries of which the nonlinear function has a jump. Denote the interiors of these configurations by  $S_1$  ( $\xi(u_h)|_{\Delta I} = 0$ ),  $S_2$  ( $\xi(u_h)|_{I_{i-1}} = 0$ ,  $\xi(u_h)|_{I_i} = 1$ ) and  $S_3$  ( $\xi(u_h)|_{I_{i-1}} = 1$ ,  $\xi(u_h)|_{I_i} = 0$ ) and the boundary between  $S_i$  and  $S_j$  may be denoted  $\Sigma_{ij}$ . By construction (see the bound (3.7)) the nonlinearity will drive any solution in  $S_2$  or  $S_3$  towards  $S_1$  and no solution whose restriction to  $\Delta I$  is in  $S_1$  can enter  $S_2$ , or  $S_3$ . This rules out the so called repulsive sliding mode that is known to cause nonuniqueness of the solution. We will not explore these issues further here but refer the interested reader to [9]. Typically in practice the system (3.1) will be discretized in time using an explicit time stepping scheme which, by definition, will produce a unique discrete solution. In the following we will prove that any solution to (3.1) will satisfy certain uniform bounds and converge to the exact solution at a certain rate.

**3.2. Maximum principles for the discrete solution.** Maximum principles give local estimates of the behavior of the solution and they rarely carry over to the discrete method. There are however stabilized methods that are specially designed to

make a discrete maximum principle hold, see for instance [23, 5] for linear convection–diffusion problems and [19, 4] for maximum principle satisfying finite element methods for conservation laws.

In [4] it was shown that the discrete equivalent of (2.3), together with energy stability of the discrete solution is sufficient to prove the convergence of the approximation sequence to the entropy solution. For our purposes herein however it is not sufficient, but we also need to prove a discrete equivalent of the bound (2.5) on the gradient. We collect the monotonicity results we need in the following lemma. For clarity of the exposition we first give the proofs in the case  $\epsilon = 0$  and then discuss how the regularization modifies the bounds.

LEMMA 3.1. *Let  $u_h$  be the solution of (3.1) either using the linear viscosity (3.2) or the nonlinear viscosity (3.5) and  $\epsilon = 0$ . Then the following bounds hold:*

$$\sup_{(x,t) \in Q} |u_h(x,t)| \leq U_0 \lesssim \max_{x \in I} |u_0(x)|, \quad (3.6)$$

$$\sup_{(x,t) \in Q} \partial_x u_h(x,t) \leq \max_{x \in I} \partial_x u_h(x,0) \lesssim \max_{x \in I} |\partial_x u_0(x)|. \quad (3.7)$$

*Proof.* The proof of (3.6) is an immediate consequence of the fact that the space discretization has the DMP-property introduced in [5].

For the case of linear artificial viscosity, first assume that for some time  $t^*$  there holds  $\max_{x \in I} |u_h(x, t^*)| \leq U_0$ , then show that this implies

$$\max_{t \geq t^*} \max_{x \in I} |u_h(x, t^*)| \leq U_0 \quad (3.8)$$

and conclude noting that the assumed inequality holds for  $t^* = 0$ , since

$$\max_{x \in I} |u_h(x, 0)| =: U_0.$$

First we compute

$$\begin{aligned} \int_I u_h \partial_x u_h v_i \, dx &= \frac{h}{3} \partial_x u_h|_{I_{i-1}}^2 + \frac{1}{6} \partial_x u_h|_{I_i}^2 \\ &\quad + \frac{h}{2} u_h(x_{i-1}) \partial_x u_h|_{I_{i-1}} + \frac{h}{2} u_h(x_i) \partial_x u_h|_{I_i} \end{aligned} \quad (3.9)$$

Assuming that  $u_h$  has a local max in  $x_i$  at  $t = t^*$  it follows that, for linear viscosity

$$\begin{aligned} h \partial_t u_h(x_i, t^*) &= -(u_h \partial_x u_h, v_i)_I - (\hat{\nu} \partial_x u_h, \partial_x v_i)_I \\ &\leq -\frac{h}{2} (U_0 - \max(|u_h(x_{i-1})|, |u_h(x_i)|)) (|\partial_x u_h|_{I_{i-1}}| + |\partial_x u_h|_{I_i}|) \leq 0. \end{aligned}$$

It follows that  $\partial_t u_h(x_i, t^*) \leq 0$  and hence the local maximum can not grow. The case of a local minimum is similar.

For nonlinear viscosity on the form (3.3), since at a local maximum  $|\partial_x u_h| = 2\{|\partial_x u_h|\}$ , we deduce that

$$\hat{\nu}(u_h)|_{I_i} = \|u_h\|_{L^\infty(I_i)} h/2 = \max(|u(x_i)|, |u(x_{i+1})|) h/2$$

and the same conclusion follows.



We will show (3.7) by first proving that the maximum gradient must be decreasing, and then applying the stability of the  $L^2$  projection. Hence it is sufficient to prove that  $\partial_t \max_i \partial_x u_h|_{I_i} \leq 0$  in  $Q$  to conclude.

We first give the proof for the linear artificial viscosity. We will prove that the discrete gradient is bounded by the gradient of the discrete initial data.

$$\sup_{(x,t) \in Q} \partial_x u_h(x,t) \leq \sup_{x \in I} \partial_x \pi_h u_0.$$

Starting from (3.1), we let  $I_i$  be any element where  $\partial_x u_h|_{I_i}$  has a local maximum, in the sense  $\partial_x u_h|_{I_i} \geq \partial_x u_h|_{I_{i \pm 1}} \geq 0$ .

$$\begin{aligned} \partial_t \partial_x u_h|_{I_i} &= -\frac{1}{h} \int_{x_{i-1}}^{x_{i+2}} u_h \partial_x u_h (v_{i+1} - v_i) \, dx - \\ &\quad \frac{1}{h} \int_{x_{i-1}}^{x_{i+2}} \hat{v} \partial_x u_h \partial_x (v_{i+1} - v_i) \, dx = T_1 + T_2. \end{aligned}$$

Decomposing the integral  $T_1$  on the contributions from  $v_i$  and  $v_{i+1}$  we have using (3.9) and some minor manipulations

$$\begin{aligned} T_1 &= -\frac{1}{6} h (\partial_x u_h|_{I_{i-1}})^2 - \frac{2}{3} h (\partial_x u_h|_{I_i})^2 - \frac{1}{6} h (\partial_x u_h|_{I_{i+1}})^2 \\ &\quad - \frac{1}{2} u_h(x_i) (\partial_x u_h|_{I_i} - \partial_x u_h|_{I_{i-1}}) - \frac{1}{2} u_h(x_{i+1}) (\partial_x u_h|_{I_{i+1}} - \partial_x u_h|_{I_i}). \end{aligned}$$

Since the derivative takes its max value in  $I_i$  we have for  $T_2$

$$\begin{aligned} T_2 &= -\frac{1}{h} \int_{x_{i-1}}^{x_{i+2}} \hat{v} \partial_x u_h \partial_x (v_{i+1} - v_i) \, dx \\ &= h^{-1} ((\hat{v}(u_h) \partial_x u_h)|_{I_{i-1}} - 2(\hat{v}(u_h) \partial_x u_h)|_{I_i} + (\hat{v}(u_h) \partial_x u_h)|_{I_{i+1}})) \leq 0. \end{aligned}$$

Collecting the above expressions and using that  $\hat{v} \geq \frac{1}{2} U_0 h$ , and we obtain

$$\begin{aligned} T_1 + T_2 &\leq -\frac{1}{6} h (\partial_x u_h|_{I_{i+1}})^2 - \frac{2}{3} h (\partial_x u_h|_{I_i})^2 - \frac{1}{6} h (\partial_x u_h|_{I_{i-1}})^2 \\ &\quad - \frac{1}{2} \underbrace{(U_0 + u_h(x_i))}_{\geq 0} \underbrace{(\partial_x u_h|_{I_i} - \partial_x u_h|_{I_{i-1}})}_{\geq 0} \\ &\quad - \frac{1}{2} \underbrace{(U_0 - u_h(x_{i+1}))}_{\geq 0} \underbrace{(\partial_x u_h|_{I_i} - \partial_x u_h|_{I_{i+1}})}_{\geq 0} \leq 0. \end{aligned}$$

This proves that  $\partial_t \max_i \partial_x u_h|_{I_i} \leq 0$  and therefore the maximum space derivative is always decreasing. The global upper bound (3.7) is immediate by the stability of the  $L^2$ -projection. Note that by the non-monotonicity of the  $L^2$ -projection there is now an absolute value on the derivative of the initial data.

In the case of the nonlinear viscosity given by (3.5) we first show that the time derivative of the gradient must be negative in cells with  $\xi(u_h) = 1$ . Then we show that any adjacent element, cannot grow either due to the design of the nonlinear switch.

In this case have after integration

$$\begin{aligned}
T_1 + T_2 = & -\frac{1}{6}h(\partial_x u_h|_{I_{i+1}})^2 - \frac{2}{3}h(\partial_x u_h|_{I_i})^2 - \frac{1}{6}h(\partial_x u_h|_{I_{i-1}})^2 \\
& - h^{-1}(\hat{\nu}_h(u_h)|_{I_i}\partial_x u_h|_{I_i} - \hat{\nu}_h(u_h)|_{I_{i-1}}\partial_x u_h|_{I_{i-1}}) \\
& - h^{-1}(\hat{\nu}_h(u_h)|_{I_i}\partial_x u_h|_{I_i} - \hat{\nu}_h(u_h)|_{I_{i+1}}\partial_x u_h|_{I_{i+1}}) \\
& - \frac{1}{2}u_h(x_i)(\partial_x u_h|_{I_i} - \partial_x u_h|_{I_{i-1}}) + \frac{1}{2}u_h(x_{i+1})(\partial_x u_h|_{I_i} - \partial_x u_h|_{I_{i+1}}). \quad (3.10)
\end{aligned}$$

First observe that the case where either  $x_i$  or  $x_{i+1}$  is a local extremum can be excluded, since then  $\nu_0|_{I_i} = \frac{1}{2}\|u_h\|_{L^\infty(I)}$  and by observing the sign of the contribution of the derivative from the neighbouring cells in the viscosity terms of line two and three of (3.10). The other terms are controlled as in the linear theory with minor modifications.

Since the gradient has a local max in  $I_i$ , in the sense that  $\xi(u_h)|_{I_i} = 1$  there holds

$$\hat{\nu}(u_h)|_{I_i} = (\max(\nu_0(u_h)|_{I_{i-1}}\frac{\partial_x u_h|_{I_{i-1}}}{\partial_x u_h|_{I_i}}, \nu_0(u_h)|_{I_{i+1}}\frac{\partial_x u_h|_{I_{i+1}}}{\partial_x u_h|_{I_i}}) + \nu_0(u_h)|_{I_i})$$

and since by construction  $\xi(u_h)|_{I_{i\pm 1}} = 0$ , we have in the neighbouring cells,

$$\hat{\nu}(u_h)|_{I_{i\pm 1}} = \nu_0(u_h)|_{I_{i\pm 1}}.$$

Using the values of  $\hat{\nu}$  the contribution from the viscous part of the differential operator may be bounded as

$$\begin{aligned}
& -h^{-1}(\hat{\nu}_h(u_h)|_{I_i}\partial_x u_h|_{I_i} - \hat{\nu}_h(u_h)|_{I_{i-1}}\partial_x u_h|_{I_{i-1}}) \\
& - h^{-1}(\hat{\nu}_h(u_h)|_{I_i}\partial_x u_h|_{I_i} - \hat{\nu}_h(u_h)|_{I_{i+1}}\partial_x u_h|_{I_{i+1}}) \\
& \leq -2\nu_0(u_h)|_{I_i}\partial_x u_h|_{I_i}. \quad (3.11)
\end{aligned}$$

For the two last terms in the right hand side of (3.10) we note that, assuming first  $u_h(x_{i+1}) \geq u_h(x_i) \geq 0$ ,

$$\begin{aligned}
& -\frac{1}{2}u_h(x_i)(\partial_x u_h|_{I_i} - \partial_x u_h|_{I_{i-1}}) + \frac{1}{2}u_h(x_{i+1})(\partial_x u_h|_{I_i} - \partial_x u_h|_{I_{i+1}}) \\
& \underbrace{\geq 0} \\
& \leq \nu_0(u_h)|_{I_i}(|\partial_x u_h|_{I_i}| + |\partial_x u_h|_{I_{i+1}}|) \quad (3.12)
\end{aligned}$$

Collecting (3.11) and (3.12), (3.10) can be upper bounded in the following fashion, recalling that  $\partial_x u_h|_{I_i} > \partial_x u_h|_{I_{i+1}} > 0$ :

$$\begin{aligned}
T_1 + T_2 \leq & -\frac{1}{6}h(\partial_x u_h|_{I_{i+1}})^2 - \frac{2}{3}h(\partial_x u_h|_{I_i})^2 - \frac{1}{6}h(\partial_x u_h|_{I_{i-1}})^2 \\
& - 2\nu_0(u_h)|_{I_i}\partial_x u_h|_{I_i} + \nu_0(u_h)|_{I_i}(|\partial_x u_h|_{I_i}| + |\partial_x u_h|_{I_{i+1}}|) \leq 0. \quad (3.13)
\end{aligned}$$

The case  $0 \geq u_h(x_{i+1}) \geq u_h(x_i)$  is similar observing that in that case the last term in the right hand side of (3.10) is negative. In case  $u_h(x_i) < 0 < u_h(x_{i+1})$  we observe that only the treatment of the first contribution of the last line of (3.10) must be modified. We note that

$$-\frac{1}{2}u_h(x_i)(\partial_x u_h|_{I_i} - \partial_x u_h|_{I_{i-1}}) \leq \frac{1}{2}h(\partial_x u_h|_{I_i})^2$$

and that this term is cancelled by the second term of the right hand side in the first line of (3.10).

Finally we must check that the gradient can not increase in any portion of the domain where the gradient is constant at the maximum value over several elements from  $I_m$  to  $I_n$  at some time  $t^*$ . First note that for all elements  $I_{m+2}, \dots, I_{n-2}$  the derivative is decreasing, since only the first three terms of the right hand side of (3.10) are non-zero. By construction  $\xi(u_h)|_{I_n} = 1$  and hence the derivative is decreasing in  $I_n$ . As a consequence the derivative in  $I_{n-1}$  is either decreasing at the time  $t^*$  or will have  $\xi|_{I_{n-1}} = 1$  at  $t^* + \varepsilon$  for all  $\varepsilon > 0$  and hence be non-increasing. Similarly for  $I_m$  and  $I_{m+1}$ , the derivatives can not grow at the same rate in both cells since then  $\xi|_{I_{m+1}} = 1$  at  $t^* + \varepsilon$  for all  $\varepsilon > 0$ , since its right hand side neighbour has decreasing space derivative. If, on the other hand the time derivative of the derivative is the largest in  $I_m$  at time  $t = t^*$  then  $\xi|_{I_m} = 1$  at  $t^* + \varepsilon$  for all  $\varepsilon > 0$ , and hence the derivative can not grow. This case of several adjacent cells over which the gradient is constant is what can give rise to the so called attractive sliding mode in the Filippov theory.  $\square$

*Remark 2.* First observe that if  $\nu_1 = 0$  it is not difficult to find  $u_h$  for which  $\partial_t \max_i \partial_x u_h|_{I_i} > 0$ , so the above technique of proof requires the contribution from  $\nu_1$ , whether or not it is really necessary in practice remains unclear.

The form of  $\nu_1$  can be simplified. Lemma 3.1 also holds for

$$\nu_1(u_h)|_{I_i} := \xi(u_h)|_{I_i} \max(\nu_0|_{I_{i-1}}, \nu_0|_{I_{i+1}}), \quad (3.14)$$

since  $\frac{\partial_x u_h|_{I_{i+1}}}{\partial_x u_h|_{I_i}} < 1$ .

It was shown in [4] that a consequence of the bound (3.1) is that the total variation of  $u_h$  diminishes. We recall the result without proof.

**COROLLARY 3.2.** *Let  $u_h$  be the solution of (3.1), then there holds, for all  $t \geq 0$*

$$TV(u_h(\cdot, t)) := \int_I |\partial_x u_h(\cdot, t)| \, dx \leq TV(u_h(\cdot, 0)).$$

**3.2.1. The effect of non-zero regularization parameter  $\epsilon$ .** In practice it may be practical to use a value on  $\epsilon$  that is related to the mesh size, in particular if implicit solvers are used. This will result in a modification of the upper bounds (3.6) and (3.7), but as we show below, the maximum principles can only be violated by an  $\mathcal{O}(\epsilon)$ .

**PROPOSITION 3.3.** *Let  $0 < \epsilon T < 1$  in (3.3), let  $u_h$  be the solution of (3.1), then there holds*

$$\|u_h(\cdot, T)\|_{L^\infty(Q)} \leq (1 + \epsilon T)U_0 \quad (3.15)$$

and

$$\max_{(x,t) \in Q} \partial_x u_h(x, t) \leq \max_{x \in I} \partial_x u_h(x, 0) + U_0(1 + \epsilon T)\epsilon T. \quad (3.16)$$

*Proof.* Assume for simplicity that  $u_h \geq 0$  and that  $u_h$  takes a global (positive) maximum in  $x_i$  that will grow with the maximum rate throughout the computation. Introduce the notation

$$g_1 := |\partial_x u_h|_{I_{i-1}}|, \quad g_2 := |\partial_x u_h|_{I_i}|.$$

Recall that at a local maximum of  $u_h$ ,  $\xi(u_h) = 0$  and therefore  $\hat{\nu}(u_h) = \max(\nu, \nu_0(u_h))$ . Assume that the maximum is taken for  $\nu_0(u_h)$ . Then by (3.9)

$$\begin{aligned}\partial_t u_h(x_i) &\leq \frac{1}{2}u_h(x_i)(g_1 + g_2) - h^{-1}\nu_h(u_h)|_{I_{i-1}}g_1 - h^{-1}\nu_h(u_h)|_{I_i}g_2 \\ &\leq \frac{1}{2}u_h(x_i)(g_1 + g_2 - \frac{(g_1 + g_2)^2}{g_1 + g_2 + \epsilon}) \leq \frac{1}{2}u_h(x_i)\epsilon.\end{aligned}$$

By Gronwall's lemma it follows that

$$u_h(x_i, T) \leq U_0 e^{\frac{1}{2}\epsilon T}.$$

Since  $e^x < 1 + x/(1 - x)$  we conclude, using the assumption that  $\epsilon T < 1$ ,

$$u_h(x_i, T) \leq U_0(1 + \epsilon T).$$

To obtain the inequality (3.16) we reason in a similar fashion starting from the equation (3.10) and using (3.15). The regularization only comes into effect at the step (3.12) and we observe that in this case

$$\begin{aligned}-\frac{1}{2}u_h(x_i)(\partial_x u_h|_{I_i} - \partial_x u_h|_{I_{i-1}}) + \frac{1}{2}u_h(x_{i+1})(\partial_x u_h|_{I_i} - \partial_x u_h|_{I_{i+1}}) \\ \leq \nu_0(u_h)|_{I_i}(|\partial_x u_h|_{I_i} + |\partial_x u_h|_{I_{i+1}} + \epsilon).\end{aligned}\quad (3.17)$$

This then leads to the bound

$$\partial_t \partial_x u_h|_{I_i} \leq -\frac{1}{6}h(\partial_x u_h|_{I_{i+1}})^2 - \frac{2}{3}h(\partial_x u_h|_{I_i})^2 - \frac{1}{6}h(\partial_x u_h|_{I_{i-1}})^2 + \frac{1}{2}u_h(x_{i+1})\epsilon.$$

Integrating in time shows that

$$\max_{(x,t) \in Q} \partial_x u_h \leq \max_{x \in I} \partial_x u_h(x, 0) + U_0(1 + \epsilon T)\epsilon T.$$

□

These perturbations of the discrete maximum principles then modifies the result (3.2) in a nontrivial way. This will not be explored herein and we will only consider the case  $\epsilon = 0$  for the stability estimates below.

**3.3. Energy stability.** Our estimates rely on stability of the numerical scheme and regularity of the dual perturbation equation. We need to control certain Sobolev norms of the discrete solution in energy type estimates similar to that of the continuous problem. The proof of the below estimates can be simplified in the linear case and the inverse estimate on the  $L^\infty$ -norm that is only valid in one dimension can then be avoided. Here we only give the proof valid both in the linear and in the nonlinear case.

**LEMMA 3.4.** *The solution  $u_h$  of the formulation (3.1) with either the linear artificial viscosity given by (3.2) or the nonlinear one of (3.5) with  $\epsilon = 0$ , satisfies the upper bounds*

$$\|u_h(T)\| + \|\hat{\nu}^{\frac{1}{2}} \partial_x u_h\|_Q \lesssim \|u_0\| \quad (3.18)$$

$$\|\partial_t u_h\|_Q \lesssim (U_0 T^{\frac{1}{2}} h^{-\frac{1}{2}} + \nu^{\frac{1}{2}}) \|\partial_x u_0\|. \quad (3.19)$$

*Proof.* The estimate (3.18) is immediate by taking  $v_h = u_h$  and noticing, by integration by parts and the periodic boundary conditions, that the nonlinear transport term vanishes. By norm equivalence and the stability of the  $L^2$ -projection

$$\|u_h(T)\| \lesssim \|u_h(T)\|_h \text{ and } \|u_h(0)\|_h \lesssim \|u_0\|.$$

The second estimate follows by taking  $v_h = \partial_t u_h$  to obtain

$$\int_0^T \|\partial_t u_h\|_h^2 dt = - \int_0^T (u_h \partial_x u_h, \partial_t u_h)_I dt - \int_0^T (\hat{\nu} \partial_x u_h, \partial_x \partial_t u_h)_I dt. \quad (3.20)$$

First note that by Corollary 3.2 and (3.6) we have, since  $TV(u_h) \leq \text{meas}(I)^{\frac{1}{2}} \|\partial_x u_h\|$ ,

$$\begin{aligned} \int_0^T (u_h \partial_x u_h, \partial_t u_h)_I dt &\leq U_0 TV(u_h(\cdot, 0)) \int_0^T \|\partial_t u_h(\cdot, t)\|_{L^\infty(I)} dt \\ &\lesssim U_0 \|\partial_x u_h(\cdot, 0)\| T^{\frac{1}{2}} h^{-\frac{1}{2}} \|\partial_t u_h\|_Q \end{aligned} \quad (3.21)$$

For the last term in the right hand side of (3.20) we observe that,

$$\begin{aligned} &\int_0^T (\hat{\nu} \partial_x u_h, \partial_x \partial_t u_h)_I dt \\ &= \int_0^T (\max(0, \hat{\nu} - \nu) \partial_x u_h, \partial_x \partial_t u_h)_I dt + \int_0^T (\nu \partial_x u_h, \partial_x \partial_t u_h)_I dt \\ &\leq TV(u_h(\cdot, 0)) \int_0^T \|\hat{\nu} \partial_x \partial_t u_h\|_{L^\infty(I)} dt + \frac{1}{2} \|\nu \partial_x u_h(\cdot, T)\|^2 - \frac{1}{2} \|\nu \partial_x u_h(\cdot, 0)\|^2 \\ &\lesssim U_0 \|\partial_x u_h(\cdot, 0)\| h^{-\frac{1}{2}} T^{\frac{1}{2}} \|\partial_t u_h\|_{L^2(Q)} \\ &\quad + \frac{1}{2} \|\nu^{\frac{1}{2}} \partial_x u_h(\cdot, T)\|^2 - \frac{1}{2} \|\nu^{\frac{1}{2}} \partial_x u_h(\cdot, 0)\|^2. \end{aligned} \quad (3.22)$$

Hence by applying (3.21) and (3.22) in the right hand side of (3.20) and norm equivalence in the left hand side of (3.20) we obtain the bound

$$\begin{aligned} \|\partial_t u_h\|_Q^2 &\leq C_q \int_0^T \|\partial_t u_h\|_h^2 dt \\ &\leq C_q^2 2U_0^2 \|\partial_x u_h(\cdot, 0)\|^2 T h^{-1} + \frac{1}{2} \|\partial_t u_h\|_{L^2(Q)}^2 + C_q^2 \frac{\nu}{2} \|\partial_x u_h(\cdot, 0)\|^2. \end{aligned}$$

The conclusion is immediate.  $\square$

**4. The linearized dual adjoint.** We introduce the linearized adjoint problem

$$\begin{aligned} -\partial_t \varphi + a(u, u_h) \partial_x \varphi - \nu \partial_{xx} \varphi &= 0 \text{ in } Q, \\ \varphi(0, t) &= \varphi(1, t) \text{ for } t \in (0, T], \\ \varphi(x, T) &= \psi(x) \text{ for } x \in I, \end{aligned} \quad (4.1)$$

where  $a(u, u_h) := (u + u_h)/2$ . The rationale for the dual adjoint is the following derivation of a perturbation equation for the functional of the error  $|(e(T), \psi)_I|$ , where

$$e(T) := u(T) - u_h(T).$$

$$\begin{aligned} |(e(T), \psi)_I| &= |(e(T), \psi)_I + \int_0^T (e, -\partial_t \varphi + a(u, u_h) \partial_x \varphi - \nu \partial_{xx} \varphi)_I \, dt| \\ &= |(e(0), \varphi(0))_I + \int_0^T (\partial_t e + \partial_x(a(u, u_h)e), \varphi)_I \, dt + \int_0^T (\nu \partial_x e, \partial_x \varphi)_I \, dt| \\ &= |(e(0), \varphi(0))_I - \int_0^T (\partial_t u_h + u_h \partial_x u_h, \varphi)_I \, dt - \int_0^T (\nu \partial_x u_h, \partial_x \varphi)_I \, dt|. \end{aligned} \quad (4.2)$$

This relation connects the error to the computational residual weighted with the solution to the adjoint problem and can lead both to a posteriori error estimates and to a priori error estimates, provided we have sufficient information on the stability properties of the numerical discretization methods and of the dual problem. The a posteriori error estimate uses techniques similar to the now classical dual weighted residual method, however in our case we can estimate the dual weights analytically, accounting for perturbations, both in the discrete and the continuous solution. Combining the a posteriori bounds with strong stability properties of the numerical method, leads to a priori upper bounds of the a posteriori quantities, showing that these must converge and in consequence that the error goes to zero. Before proceeding with this analysis we derive an a priori estimate for the derivatives of the dual adjoint (4.1).

**4.1. Wellposedness and stability.** Since  $a(u, u_h) \in W^{1,\infty}(I)$  the problem (4.1) has a unique solution and one may show that it satisfies the maximum principle

$$\max_{(x,t) \in Q} |\varphi(x,t)| \leq \max_{x \in I} |\psi(x)|.$$

The following stability estimate follows easily by standard energy methods

LEMMA 4.1. *Let  $\varphi$  be the solution to (4.1) then there holds*

$$\sup_{t \in (0,T)} \|\partial_x \varphi(\cdot, t)\|^2 + \nu \|\partial_{xx} \varphi\|_Q^2 \lesssim \exp(D_0 T) \|\partial_x \psi\|^2. \quad (4.3)$$

*Proof.* Multiply the equation (4.1) by  $-\partial_{xx} \varphi$  and integrate over  $(t, T)$

$$\|\partial_x \varphi(\cdot, t)\|^2 + 2 \int_t^T \nu \|\partial_{xx} \varphi\|^2 \, dt = \|\partial_x \psi\|^2 - 2 \int_t^T (a(u, u_h) \partial_x \varphi, \partial_{xx} \varphi) \, dt. \quad (4.4)$$

Note that, by an integration by parts and the maximum principles (2.3) and (3.7)

$$\begin{aligned} 2 \int_t^T (a(u, u_h) \partial_x \varphi, \partial_{xx} \varphi)_I \, dt &= - \int_t^T (\partial_x a(u, u_h) \partial_x \varphi, \partial_x \varphi) \, dt \\ &\geq -D_0 \int_t^T \|\partial_x \varphi\|^2 \, dt. \end{aligned}$$

The result in  $L^\infty(0, T; L^2(I))$  follows from the Gronwall's lemma and taking the supremum over  $t \in (0, T)$  of the resulting expression

$$\|\partial_x \varphi(\cdot, t)\|^2 \lesssim \exp(D_0 t) \|\partial_x \psi\|^2.$$

The result for the second derivatives then follows by using this expression to bound the right hand side of (4.4)

$$\begin{aligned} 2 \int_0^T \nu \|\partial_{xx} \varphi\|^2 dt &\leq \|\partial_x \psi\|^2 + \int_0^T (\partial_x a(u, u_h) \partial_x \varphi, \partial_x \varphi) dt \\ &\lesssim \|\partial_x \psi\|^2 \left(1 + D_0 \int_0^T \exp(D_0 t) dt\right) = \exp(D_0 T) \|\partial_x \psi\|^2. \end{aligned}$$

□

**5. Error estimates for filtered quantities.** We will consider the differential filter defined in (1.2), where  $\delta$  denotes a filter width to be specified. The norm associated to the differential filter is given by

$$\|\tilde{u}\|_\delta := (\|\delta \partial_x \tilde{u}\|^2 + \|\tilde{u}\|^2)^{\frac{1}{2}}.$$

We introduce the filtered error  $\tilde{e} := \tilde{u} - \tilde{u}_h$ , where  $\tilde{u}$  and  $\tilde{u}_h$  denote the filtered exact and approximate solutions respectively obtained by solving (1.2) with  $u$  and  $u_h$  as right hand side. The analysis uses the stability properties of the adjoint perturbation equation (Lemma 4.1) and the stability properties of the discrete problem (Lemma 3.4) to derive first a posteriori error bounds for the filtered quantities and then a priori bounds by upper bounding the a posteriori residuals, by a priori quantities.

**THEOREM 5.1.** *Let  $u$  be the solution of (1.1),  $u_h$  be the solution of (3.1). Then the following holds:*

- *A posteriori upper bound*

$$\begin{aligned} \|\tilde{e}(T)\|_\delta &\lesssim \exp(D_0 T) \left( \frac{h}{\delta^2} \right)^{\frac{1}{2}} \left( h^{\frac{1}{2}} \|(u - u_h)(0)\| + h^{\frac{1}{2}} \int_0^T \inf_{v_h \in V_h} \|v_h + u_h \partial_x u_h\| dt \right. \\ &\quad \left. + h^{\frac{3}{2}} \int_0^T \|\partial_x \partial_t u_h\| dt + \int_0^T \|\max(0, \hat{\nu} - \nu)^{\frac{1}{2}} \partial_x u_h\| dt \right. \\ &\quad \left. + h \left( \int_0^T \nu \|\partial_x u_h\|_N^2 dt \right)^{\frac{1}{2}} \right), \quad (5.1) \end{aligned}$$

where

$$\|\partial_x u_h\|_N := \left( \sum_{i=0}^{N-1} (\partial_x u_h(x_i)|_{I_{i+1}} - \partial_x u_h(x_i)|_{I_i})^2 \right)^{\frac{1}{2}},$$

with  $I_N$  identified with  $I_0$  by periodicity.

- *A priori upper bound*

$$\|\tilde{e}\|_\delta \lesssim \exp(D_0 T) \left( \frac{h}{\delta^2} \right)^{\frac{1}{2}} \left( \left( h^{\frac{1}{2}} + U_0^{\frac{1}{2}} \sqrt{T} \right) \|u_0\| + (TU_0 + h^{\frac{1}{2}} \nu^{\frac{1}{2}}) \|\partial_x u_0\| \right). \quad (5.2)$$

*Proof.* Let  $\psi = \tilde{e}(T)$  in the definition (4.1) of the dual adjoint problem. Using the design of the dual problem and the definition of  $\tilde{e}$  we have by the relation (4.2)

$$\begin{aligned} \|\tilde{e}(T)\|_\delta^2 &= (\delta \partial_x \tilde{e}(T), \partial_x \tilde{e}(T))_I + (\tilde{e}(T), \tilde{e}(T))_I = (e(T), \tilde{e}(T))_I \\ &= (e(0), \varphi(0))_I - \int_0^T (\partial_t u_h + u_h \partial_x u_h, \varphi)_I dt - \int_0^T (\nu \partial_x u_h, \partial_x \varphi)_I dt. \end{aligned}$$

Taking  $v_h = \pi_h \varphi$ , with  $\pi_h$  denoting the standard  $L^2$ -projection, in (3.1) and adding to the above expression yields

$$\begin{aligned}
\|\tilde{e}\|_\delta^2 &= (e(0), \varphi(0))_I - \int_0^T (\partial_t u_h + u_h \partial_x u_h, \varphi)_I \, dt - \int_0^T (\nu \partial_x u_h, \partial_x \varphi)_I \, dt \\
&\quad + \int_0^T (\partial_t u_h, \pi_h \varphi)_h \, dt + \int_0^T (u_h \partial_x u_h, \pi_h \varphi)_I \, dt + \int_0^T (\hat{\nu} \partial_x u_h, \partial_x \pi_h \varphi)_I \, dt \\
&\quad = \underbrace{(e(0), \varphi(0))_I}_{T_0} - \underbrace{\int_0^T (\partial_t u_h + u_h \partial_x u_h, \varphi - \pi_h \varphi)_I \, dt}_{T_1} \\
&\quad - \underbrace{\int_0^T ((\partial_t u_h, \pi_h \varphi)_I - (\partial_t u_h, \pi_h \varphi)_h) \, dt}_{T_2} - \underbrace{\int_0^T (\nu \partial_x u_h, \partial_x (\varphi - \pi_h \varphi))_I \, dt}_{T_3} \\
&\quad \quad \quad + \underbrace{\int_0^T (\max(0, \hat{\nu}(u_h) - \nu) \partial_x u_h, \partial_x \pi_h \varphi)_I \, dt}_{T_4}.
\end{aligned}$$

Now consider the terms  $T_0$  to  $T_4$  term by term. First use the orthogonality  $(e_0, v_h)_I = 0$  for all  $v_h \in V_h$ ,

$$T_0 = (e(0), \varphi - \pi_h \varphi)_I \lesssim \|e_0\| h \sup_{t \in (0, T)} \|\partial_x \varphi(t)\|.$$

Similarly for all  $w_h \in V_h$  there holds

$$\begin{aligned}
T_1 &= - \int_0^T (w_h + u_h \partial_x u_h, \varphi - \pi_h \varphi)_I \, dt \\
&\leq \int_0^T \|w_h + u_h \partial_x u_h\| \, dt \sup_{t \in (0, T)} \|(\varphi - \pi_h \varphi)(t)\|
\end{aligned}$$

and hence

$$T_1 \lesssim h \int_0^T \inf_{v_h \in V_h} \|v_h + u_h \partial_x u_h\| \, dt \sup_{t \in (0, T)} \|\partial_x \varphi(t)\|.$$

Let  $\mathcal{I}_h$  denote the standard Lagrange interpolant. By the definition of the discrete  $L^2$ -inner product  $(\cdot, \cdot)_h$  there holds

$$\begin{aligned}
T_2 &= - \int_0^T \int_I (\partial_t u_h \pi_h \varphi - \mathcal{I}_h(\partial_t u_h \pi_h \varphi)) \, dx \, dt \\
&\leq \int_0^T \int_I h^2 |\partial_x \partial_t u_h \partial_x \pi_h \varphi| \, dx \, dt \lesssim \int_0^T h^2 \|\partial_x \partial_t u_h\| \|\partial_x \pi_h \varphi\| \, dt \\
&\lesssim \int_0^T h^2 \|\partial_x \partial_t u_h\| \, dt \sup_{t \in (0, T)} \|\partial_x \varphi(t)\|.
\end{aligned}$$



For  $T_3$  we have after an integration by parts and using a trace inequality followed by approximation

$$\begin{aligned}
T_3 &= \int_0^T \sum_{i=0}^{N-1} \nu (\partial_x u_h(x_i)|_{I_{i+1}} - \partial_x u_h(x_i)|_{I_i}) (\varphi(x_i) - \pi_h \varphi(x_i)) \, dt \\
&\lesssim \int_0^T \nu \|[\partial_x u_h]\|_N (h^{-\frac{1}{2}} \|\varphi - \pi_h \varphi\| + h^{\frac{1}{2}} \|\partial_x(\varphi - \pi_h \varphi)\|) \, dt \\
&\lesssim \left( \int_0^T \nu \|[\partial_x u_h]\|_N^2 \, dt \right)^{\frac{1}{2}} h^{\frac{3}{2}} \|\nu^{\frac{1}{2}} \partial_{xx} \varphi\|_Q
\end{aligned}$$

Finally the non-consistent artificial viscosity term is controlled using the Cauchy-Schwarz inequality and the  $H^1$ -stability of the  $L^2$ -projection  $\|\partial_x \pi_h \varphi\| \lesssim \|\partial_x \varphi\|$

$$\begin{aligned}
T_4 &\leq \max_i \hat{\nu}|_{I_i} \int_0^T \|\max(0, \hat{\nu} - \nu)^{\frac{1}{2}} \partial_x u_h\| \, dt \sup_{t \in (0, T)} \|\partial_x \varphi(t)\| \\
&\lesssim (U_0 h)^{\frac{1}{2}} \int_0^T \|\max(0, \hat{\nu} - \nu)^{\frac{1}{2}} \partial_x u_h\| \, dt \sup_{t \in (0, T)} \|\partial_x \varphi(t)\|.
\end{aligned}$$

Collecting the bounds for  $T_0 - T_4$  and using the stability estimate (4.3) we have

$$\begin{aligned}
\|\tilde{e}(T)\|_\delta^2 &\lesssim \exp(D_0 T) \left( \frac{h}{\delta^2} \right)^{\frac{1}{2}} \left( h^{\frac{1}{2}} \|e(0)\| + h^{\frac{1}{2}} \int_0^T \inf_{v_h \in V_h} \|v_h + u_h \partial_x u_h\| \, dt \right. \\
&\quad + U_0^{\frac{1}{2}} \int_0^T \|\max(0, \hat{\nu} - \nu)^{\frac{1}{2}} \partial_x u_h\| \, dt + h^{\frac{3}{2}} \int_0^T \|\partial_x \partial_t u_h\| \, dt \\
&\quad \left. + h \left( \int_0^T \nu \|[\partial_x u_h]\|_N^2 \, dt \right)^{\frac{1}{2}} \right) \|\tilde{e}(T)\|_\delta \quad (5.3)
\end{aligned}$$

from which (5.1) follows.

The a priori error estimate now follows by using discrete stability to bound the residuals. Since we do not assume any regularity of the exact solution we can not assume that any stronger bounds hold. Note that by a well known discrete interpolation estimate there holds,

$$\inf_{v_h \in V_h} \|h^{\frac{1}{2}}(v_h - u_h \partial_x u_h)\| \lesssim h \| [u_h \partial_x u_h] \|_N.$$

Using that

$$\begin{aligned}
h[u_h \partial_x u_h]|_{x_i} &\leq h|u_h(x_i)| \left( \frac{\|[\partial_x u_h]\|}{2\{[\partial_x u_h]\} + \epsilon} \right) |_{x_i} (|\partial_x u_h|_{I_{i-1}} + |\partial_x u_h|_{I_i} + \epsilon) \\
&\leq (\hat{\nu}(u_h) \partial_x u_h)|_{I_{i-1}} + (\hat{\nu}(u_h) \partial_x u_h)|_{I_i} + hU_0 \epsilon
\end{aligned}$$

we may deduce

$$\inf_{v_h \in V_h} \|h^{\frac{1}{2}}(v_h - u_h \partial_x u_h)\| \lesssim U_0^{\frac{1}{2}} \|\hat{\nu}^{\frac{1}{2}} \partial_x u_h\| + h^{\frac{1}{2}} U_0 \epsilon.$$

In the linear case the inequality is trivial by taking  $v_h = 0$  and using (3.6).

Then use the Cauchy-Schwarz inequality in time for the two terms of the second line of (5.3), an inverse inequality for the second term in the second line and a trace inequality for the last term of (5.3).

$$\begin{aligned} |||\tilde{e}(T)|||_\delta &\lesssim \exp(D_0 T) \left( \frac{h}{\delta^2} \right)^{\frac{1}{2}} \left( h^{\frac{1}{2}} \|e(0)\| + (h^{\frac{1}{2}} + U_0^{\frac{1}{2}} \sqrt{T}) \|\hat{\nu}^{\frac{1}{2}} \partial_x u_h\|_Q \right. \\ &\quad \left. + h^{\frac{1}{2}} \sqrt{T} \|\partial_t u_h\|_Q + T^{\frac{1}{2}} h^{\frac{1}{2}} U_0 \epsilon \right). \end{aligned}$$

We conclude by applying the stability estimates and (3.18) and (3.19) leading to

$$\begin{aligned} |||\tilde{e}|||_\delta &\lesssim \exp(D_0 T) \left( \frac{h}{\delta^2} \right)^{\frac{1}{2}} \left( (h^{\frac{1}{2}} + U_0^{\frac{1}{2}} \sqrt{T}) \|u_0\| \right. \\ &\quad \left. + (TU_0 + h^{\frac{1}{2}} \nu^{\frac{1}{2}}) \|\partial_x u_0\| + T^{\frac{1}{2}} h^{\frac{1}{2}} U_0 \epsilon \right). \end{aligned}$$

□

Observe that the above estimate is independent both of the regularity of the exact solution and of the flow regime.

*Remark 3.* Note that if instead the initial data  $u_h(x, 0)$  is chosen as the nodal interpolant of  $u_0$ ,  $\mathcal{I}_h u_0$ , we may define  $U_0 = \max_{x \in I} |u_0(x)|$  and  $D_0 = \max_{x \in I} |\partial_x u_0(x)|$ . On the other hand we can no longer use  $L^2$ -orthogonality in the upper bound for  $T_0$ . It appears that in that case we must use the maximum principle of the dual problem to obtain

$$\begin{aligned} T_0 = (e(0), \varphi)_I &\lesssim \|e(0)\|_{L^1(I)} \|\varphi(0)\|_{L^\infty(I)} \\ &\leq \|e(0)\|_{L^1(I)} \|\Psi\|_{L^\infty(I)} \lesssim \delta^{-1} \|e(0)\|_{L^1(I)} |||\tilde{e}|||_\delta \\ &\lesssim \left( \frac{h^s}{\delta} \right) \|\partial_x^s u_0\|_{L^1(I)} |||\tilde{e}|||_\delta, \quad s = 1, 2. \end{aligned}$$

The global convergence order will be the same, but it appears that the error contribution from the initial data will be larger and the factor  $\|\partial_x^s u_0\|_{L^1(I)}$  must be added to the right hand side of (5.2). Another downside to this approach is that it only works in one space dimension, whereas before only the energy stability estimates of Lemma 3.4 used one dimensional inverse inequalities.

*Remark 4.* We have kept the dependence on the regularization parameter  $\epsilon$  in the above proof. This shows the effect of regularization on the computational error under the assumption that (3.2) still holds under regularization.

*Remark 5.* Since all these estimates are independent of  $\nu$  they are also valid for the purely hyperbolic case, with  $u$  the entropy solution.

**6. Numerical examples.** In this section we will study two numerical examples computed with  $\nu = 0$ . We first consider a problem with smooth initial data

$$u_0 = \frac{1}{2}(\cos(\pi x) + 1).$$

We compute the solution at  $T = 0.5$ , before shock formation and compute the exact solution on a mesh with 6400 mesh points using fixed point iteration. The initial data and the final solutions are given in Figure 6.1. In Table 6.1 errors in several

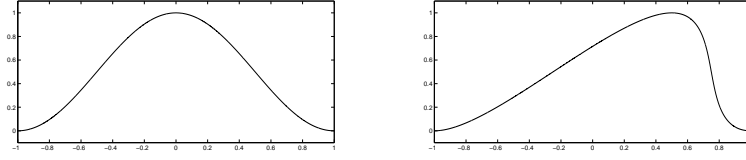


FIG. 6.1. *Left smooth initial condition; right solution at  $T = 0.5$*

N	$\ u - u_h\ _{L^1(I)}$	$\ u - u_h\ _{L^2(I)}$	$\ \tilde{e}\ _1$	$\ \tilde{e}\ _h$
100	$2.5 \cdot 10^{-3}$	$3.6 \cdot 10^{-3}$	$3.0 \cdot 10^{-4}$	$3.2 \cdot 10^{-3}$
200	$6.7 \cdot 10^{-4}$ (1.9)	$1.0 \cdot 10^{-3}$ (1.8)	$7.0 \cdot 10^{-5}$ (2.1)	$9.5 \cdot 10^{-4}$ (1.8)
400	$1.8 \cdot 10^{-4}$ (1.9)	$3.0 \cdot 10^{-4}$ (1.7)	$1.7 \cdot 10^{-5}$ (2.0)	$2.9 \cdot 10^{-5}$ (1.7)
800	$4.6 \cdot 10^{-5}$ (2.0)	$8.9 \cdot 10^{-5}$ (1.8)	$4.2 \cdot 10^{-6}$ (2.0)	$8.7 \cdot 10^{-6}$ (1.7)

TABLE 6.1  
 $\epsilon = O(10^{-16})$ , *smooth solution*

different norms is presented on four consecutive meshes. Here  $\epsilon = 0$  to machine precision. Experimental convergence rates are given in parenthesis. In the following table (Table 6.2) we present the same results for  $\epsilon = h$ . The results are similar, with the difference that the regularized method gives second order convergence in all norms, whereas the one without regularization exhibits a slight reduction in the order in the  $L^2$ -norm. This is not surprising since formally the order of the method is  $h^{\frac{3}{2}}$  and the unregularized method adds nonconsistent first order viscosity at local extrema.

Now we consider a problem with non-smooth solution. The initial data and final time exact solution is given in Figure 6.2. We compute the solution at  $T = 0.5$  when the shock has formed. The exact solution is computed using the method of characteristics on a mesh with 12800 elements. We present tables with the same errors as in the previous case for the method without (Table 6.3) and with (Table 6.4) regularization. For this case there is even less difference between the two cases. We observe first order convergence for the  $L^1$ -error and the  $H^{-1}$ -norm error and  $1/2$ -order convergence in the  $L^2$ -norm. The computations clearly show how the weaker norm behaves either as an  $L^1$ -norm for  $\delta = 1$  or an  $L^2$ -norm for  $\delta = h$ . Intermediate values of  $\delta$  appears to interpolate between these two norms. This indicates that the principle of our estimate, with the order depending on how  $\delta$  is chosen with respect to  $h$  is correct. A superconvergence of approximately half an order is observed for all the computations with the nonsmooth solution, compared to what is predicted by theory.

**7. Discussion.** Above we proved that the error in the filtered finite element solution to the Burgers' equation allows for convergence proofs with order  $\mathcal{O}(h^\epsilon)$ ,  $\epsilon \in (0, \frac{1}{2}]$ , for filter width  $\delta = h^{(1-2\epsilon)/2}$  and moderate constants, even though the analysis for the unfiltered quantity leads to error bounds with huge exponential constants. The results hold both for linear artificial viscosity and nonlinear viscosity of shock capturing type. To our best knowledge this is the first time it has been rigorously proven that stabilized finite element methods combined with filtering of the final result improves computability.

## REFERENCES

N	$\ u - u_h\ _{L^1(I)}$	$\ u - u_h\ _{L^2(I)}$	$\ \tilde{e}\ _1$	$\ \tilde{e}\ _h$
100	$1.9 \cdot 10^{-3}$	$3.0 \cdot 10^{-3}$	$2.3 \cdot 10^{-4}$	$2.6 \cdot 10^{-3}$
200	$4.7 \cdot 10^{-4}$ (2.0)	$7.7 \cdot 10^{-4}$ (2.0)	$5.5 \cdot 10^{-5}$ (2.1)	$7.1 \cdot 10^{-4}$ (1.9)
400	$1.2 \cdot 10^{-4}$ (2.0)	$2.1 \cdot 10^{-4}$ (1.9)	$1.3 \cdot 10^{-5}$ (2.1)	$1.9 \cdot 10^{-4}$ (1.8)
800	$3.0 \cdot 10^{-5}$ (2.0)	$5.5 \cdot 10^{-5}$ (1.9)	$3.3 \cdot 10^{-6}$ (2.0)	$5.3 \cdot 10^{-5}$ (1.8)

TABLE 6.2  
 $\epsilon = h$ , smooth solution

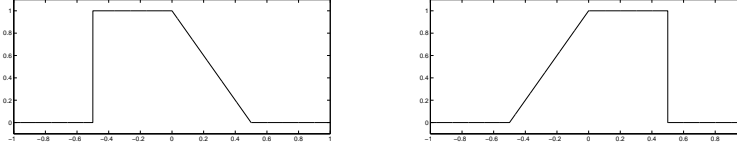


FIG. 6.2. Left nonsmooth initial condition; right solution at  $T = 0.5$

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N	$\ u - u_h\ _{L^1(I)}$	$\ u - u_h\ _{L^2(I)}$	$\ \tilde{e}\ _1$	$\ \tilde{e}\ _h$
100	0.036	0.071	$6.4 \cdot 10^{-3}$	0.038
200	0.018 (1.0)	0.049 (0.5)	$3.2 \cdot 10^{-3}$ (1.0)	0.024 (0.7)
400	$9.4 \cdot 10^{-3}$ (0.9)	0.034 (0.5)	$1.6 \cdot 10^{-3}$ (1.0)	0.016 (0.6)
800	$4.7 \cdot 10^{-3}$ (1.0)	0.023 (0.6)	$7.9 \cdot 10^{-4}$ (1.0)	0.011 (0.5)

TABLE 6.3  
 $\epsilon = O(10^{-16})$ , nonsmooth solution

N	$\ u - u_h\ _{L^1(I)}$	$\ u - u_h\ _{L^2(I)}$	$\ \tilde{e}\ _1$	$\ \tilde{e}\ _h$
100	0.035	0.070	$6.3 \cdot 10^{-3}$	0.037
200	0.018 (1.0)	0.048 (0.5)	$3.2 \cdot 10^{-3}$ (1.0)	0.024 (0.6)
400	$9.1 \cdot 10^{-3}$ (1.0)	0.033 (0.5)	$1.6 \cdot 10^{-3}$ (1.0)	0.016 (0.6)
800	$4.6 \cdot 10^{-3}$ (1.0)	0.023 (0.5)	$7.9 \cdot 10^{-4}$ (1.0)	0.010 (0.7)

TABLE 6.4  
 $\epsilon = h$ , nonsmooth solution

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